Finally, tables and formulæ are given by which the resistance of, or the loss through any new cable coated with gutta percha, may be at least approximately estimated:—

TABLE I.

Specific Resistance in Thomson's Units of the Red Sea Covering at various Temperatures.

Temperature.	Zinc to cable.		Copper to cable.	
	After electrification for one minute.	After electrification for five minutes.	After electrification for one minute.	After electrification for five minutes.
60°	2162×10^{17}	3330×10^{17}	2239×10^{17}	3405×10^{17}
65	1810 × ,,	2947 × ,,	1720 × ,,	2770 × ,,
70	1460 × ,,	2378 × ,,	1318 × ,,	2239 × ,,
75	1160 × ,,	1753 × ,,	1000 × ,,	1739 × ,,

TABLE II.

Specific Resistance in Thomson's Units of pure Gutta Percha at various Temperatures.

Temperature.	Zinc to cable.		Copper to cable.	
	After electrification for one minute.	After electrification for five minutes.	After electrification for one minute.	After electrification for five minutes.
50	4113×10^{17}	5663×10^{17}	4113×10^{17}	5663×10^{17}
55	2917 × ,,	3636 × ,,	2917 × ,,	3636 × ,,
60	2163 × ,,	2549 × ,,	2163 × ,,	2549 × ,,
65	1634 × ,,	1858 × ,,	1634 × ,,	1858 × ,,
70	1162 × ,,	1291 × ,,	1193 × ,,	1291 × ,,
75	805 × ,,	877 × ,,	796 × ,,	866 × ,,
80	566 × ,,	613 × ,,	548 × ,,	591 × ,,

III. "On Scalar and Clinant Algebraical Coordinate Geometry, introducing a new and more general Theory of Analytical Geometry, including the received as a particular case, and explaining 'imaginary points,' 'intersections,' and 'lines.'" By ALEXANDER J. ELLIS, Esq., B.A., F.C.P.S. Communicated by ARCHIBALD SMITH, Esq. Received February 16, 1860.

(Abstract.)

Scalar Plane Geometry.—With O as a centre describe a circle with a radius equal to the unit of length. Let OA, OB be any two

of its unit radii, termed 'coordinate axes.' From any point P in the plane AOB draw PM parallel to BO, so as to cut OA, produced either way if necessary, in M. Then there will exist some 'scalars' ('real' or 'possible quantities') u, v such that $\mathrm{OM}=u\,.\,\mathrm{OA}$, and $\mathrm{MP}=v\,.\,\mathrm{OB}$, all lines being considered in respect both to magnitude and direction. Hence OP, which is the 'appense' or 'geometrical sum' of OM and MP, or $=\mathrm{OM}+\mathrm{MP}$, will $=u\,.\,\mathrm{OA}+v\,.\,\mathrm{OB}$. By varying the values of the 'coordinate scalars' u, v, P may be made to assume any position whatever on the plane of AOB. The angle AOB may be taken at pleasure, but greater symmetry is secured by choosing OI and OJ as coordinate axes, where IOJ is a right angle described in the right-handed direction. If any number of lines OP, OQ, OR, &c., be thus represented, the lengths of the lines PQ, QR, &c., and the sines and cosines of the angles IOP, POQ, QOR, &c., can be immediately furnished in terms of the unit of length and the coordinate scalars.

If $\mathrm{OP}=x\,.\,\mathrm{OI}+y\,.\,\mathrm{OJ}$, and any relation be assigned between the values of x and y, such as $y=fx$ or $\phi\,(x, y)=0$, then the possible positions of P are limited to those in which for any scalar value of x there exists a corresponding scalar value of y. The ensemble of all such positions of P constitutes the '*locus*' of the *two* equations, viz. the 'concrete equation' $\mathrm{OP}=x\,.\,\mathrm{OI}+y\,.\,\mathrm{OJ}$, and the 'abstract equation' $y=f\,.\,x$. The peculiarity of the present theory consists in the recognition of these *two* equations to a curve, of which the ordinary theory only furnishes the latter, and inefficiently replaces the former by some convention respecting the use of the letters, whereby the coordinates themselves are not made a part of the calculation.

A variation in either of these two equations will occasion a difference either in the form or position of their locus. If the abstract equation be $y=ax+b$, where a and b are any scalars, the concrete equation $\mathrm{OP}=x\,.\,\mathrm{OI}+y\,.\,\mathrm{OJ}$ becomes $\mathrm{OP}=x\,.\,(\mathrm{OI}+a\,.\,\mathrm{OJ})+b\,.\,\mathrm{OJ}$, which shows that OP is the appense of a constant line, and a line in a constant direction, and hence its extremity P must lie on a line in that direction drawn through the extremity of the constant line. Also, since the length of OP is $\sqrt{(x^2+y^2)}$ times the length of OI, the locus of the two equations

$$\mathrm{OP}=x\,.\,\mathrm{OI}+y\,.\,\mathrm{OJ} \text{ and } x^2+y^2=a^2$$

must be a circle. From these equations all the usual theory of the straight line and circle may be readily deduced, and all ambiguity respecting the representation of direction by the signs (+) and (—) may be removed. Thus if the loci of

$$\mathrm{OP}=x\,.\,\mathrm{OI}+y\,.\,\mathrm{OJ},\qquad y=ax+b,$$

and

$$\mathrm{OP}'=x'\,.\,\mathrm{OI}+y'\,.\,\mathrm{OJ},\qquad ay'+x'=0$$

intersect in the point Q, and OQ_1 be the unit radius on the J side of OI (produced both ways), determined by the equation

$$\sqrt{(1+a^2)}\,.\,\mathrm{OQ}_1=-a\,.\,\mathrm{OI}+\mathrm{OJ},$$

then $\sqrt{(1+a^2)}\,.\,\mathrm{OQ}$ will $=b\,\mathrm{OQ}_1$. And a line drawn from the locus of P parallel to OQ to pass through the point X, where $\mathrm{OX}=x_1\,.\,\mathrm{OI}+y_1\,.\,\mathrm{OJ}$, will be represented in magnitude and direction by

$$\frac{y_1-ax_1-b}{\sqrt{(1+a^2)}}\,.\,\mathrm{OQ}_1.$$

From this result the usual theory of anharmonic ratios is immediately deducible without any fresh 'convention' respecting the signs (+) and (—).

As every locus has *two* equations, each equation requires separate consideration. The investigations concerning the abstract equation remain nearly the same as in the usual theory. When the abstract equation is given indirectly by the elimination of two constants between three equations, the result corresponds to the locus of the intersection of two curves varying according to a known law, 'coordinates proper,' leading in its simplest forms, first, to Descartes' original conception of curves generated by the intersection of straight lines moving according to a given law parallel to two given straight lines, and secondly, to Plücker's 'point coordinates.' The true relation of Plücker's 'line coordinates' to the ordinary system is immediately apparent on comparing the two sets of equations:

(1)	Concrete	$\mathrm{OP}=x\,.\,\mathrm{OI}+y\,.\,\mathrm{OJ}$
	Abstract	$\mathrm{F}(x, y)=0,$
(2)	First abstract	$y=ax+b$
	Second abstract	$\mathrm{F}(a, b)=0.$

The second set of equations determines a curve by supposing a and b to vary, then eliminating a and b, and referring the ultimate abstract

equation to the concrete equation (1). On comparing these two sets of equations, we see that x and y in the first are involved in precisely the same manner as a and b in the second, so that if any equation $F(u, v)=0$ were given, it might determine one or the other by precisely the same algebra, according as x, y or a, b were substituted for u, v. Whence flow Plücker's theories of collineation and reciprocity.

The investigations respecting the concrete equation, on the other hand, are altogether new. The most general form of the two equations, is

$$\text{Concrete, } \mathrm{OR}=f_1(x, y)\,.\,\mathrm{OA}+f_2(x, y)\,.\,\mathrm{OB}$$
$$\text{Abstract, } \phi(x, y)=0,$$

which will clearly determine a curve as definitely as before. If in lieu of the abstract equation $\phi(x, y)=0$, we were given the locus of the two equations

$$\mathrm{OM}=x\,.\,\mathrm{OA}+y\,.\,\mathrm{OB}, \quad \phi(x, y)=0,$$

and from any point M in this curve we drew MN parallel to OB, cutting OA produced in N, so that $\mathrm{ON}=x\,.\,\mathrm{OA}$, $\mathrm{NM}=y\,.\,\mathrm{OB}$, we could find x and y from this curve, and consequently form

$$\mathrm{OL}=f_1(x, y)\,.\,\mathrm{OA}, \text{ and } \mathrm{LR}=f_2(x, y)\,.\,\mathrm{OB},$$

and by this means determine the point R, where $\mathrm{OR}=\mathrm{OL}+\mathrm{LR}$, in the locus of the general equations, corresponding to the point M in the particular locus. The general form is therefore the algebraical expression of a curve formed from another curve by means of operations performed on the coordinates of the points in the latter, as when an ellipse is formed from a circle by altering the ordinates in a constant ratio.

The algebraical treatment of this case consists in putting

$$p=f_2(x, y),\ q=f_2(x, y),$$

and between these equations and $\phi(x, y)=0$, eliminating x and y, to find $\psi(p, q)=0$. The locus is then reduced to that of

$$\mathrm{OR}=p\,.\,\mathrm{OI}+q\,.\,\mathrm{OJ},\ \psi(p, q)=0,$$

which is the ordinary simple case. But the whole of this latter locus does not in all cases correspond to the locus of the general equations, because not only x and y, but also p or $f_1(x, y)$, and q or

$f_2(x, y)$ must be all scalar. Thus, in the case of the parabola derived from a circle, by substituting for the ordinate of a point in the latter its distance from a known point in the circumference of the same, we know that it is impossible to derive more of the parabola than can be obtained by taking the diameter of the circle as the ordinate. The algebraical process gives first

$$(1)\quad \mathrm{OR}=x\,.\,\mathrm{OI}+\surd(x^2+y^2)\,.\,\mathrm{OJ},\qquad y^2+x^2=2ax;$$

and then, putting $p=x$, $q=+\surd(x^2+y^2)$, we find

$$(2)\quad \mathrm{OR}=p\,.\,\mathrm{OI}+q\,.\,\mathrm{OJ},\ q^2=2ap.$$

In this case q is always positive and $=+\surd(2ax)$. But x, and therefore p, always lies between the limits 0 and $2a$, and hence q must lie between the same limits. Consequently the only part of the curve (2) represented by the equations (1) is the semi-parabola contained between the origin and the ordinate $2a\,.\,\mathrm{OJ}$.

This general form of the concrete equation, therefore, furnishes an elementary method of representing curves or parts of curves. Thus

$$\mathrm{OR}=(h+a\cos\alpha+x\cos\alpha)\,.\,\mathrm{OI}+\big(k(+a\sin\alpha+x\sin\alpha)\big)\,.\,\mathrm{OJ}$$
$$x^2+y^2=a^2,$$

are the equations of a line

$$\mathrm{HK}=2a\,(\cos\alpha\,.\,\mathrm{OI}+\sin\alpha\,.\,\mathrm{OJ}),$$

and having one of its extremities determined by the equation

$$\mathrm{OH}=h\,.\,\mathrm{OI}+k\,.\,\mathrm{OJ},$$

so that its length is $2a$ times that of OI and the angle $(\mathrm{OI}, \mathrm{HK})=\alpha$.

To determine the intersections of two such finite curves, given by the equations

$$\mathrm{OP}=f_1(x, y)\,.\,\mathrm{OI}+f_2(x, y)\,.\,\mathrm{OJ},\quad \phi(x, y)=0,$$

and $$\mathrm{OP}'=f_1'(x', y')\,.\,\mathrm{OI}+f_2'(x', y')\,.\,\mathrm{OJ},\quad \phi'(x', y')=0,$$

we have $\mathrm{OP}=\mathrm{OP}'$, and consequently $f_1=f_1'$ and $f_2=f_2'$, which, with $\phi=0$ and $\phi'=0$, give four equations to determine x, y, x', y'. The curves will, however, not intersect, unless not only four such scalars exist, but they make f_1, f_2, f_1', f_2' all scalar.

The transformation of coordinates may now be investigated more generally in the form of the two problems: 'given a change in the concrete (or abstract) equation, to find the corresponding change in

the abstract (or concrete) equation respectively, in order that the locus may remain unaltered.' The ordinary theory only comprehends an exceedingly simple instance of the first problem. The second is indeterminate so far as the representation of portions of curves is concerned, so that any abstract equation may, by the help of a properly selected concrete equation, represent any curve whatever.

A curve by which the scalar value of y is exhibited corresponding to any scalar value of x in the equation $\phi(x, y)=0$, and which in this simple case is furnished by the locus of the two equations

$$\mathrm{OP}=x\,.\,\mathrm{OI}+y\,.\,\mathrm{OJ},\quad \phi(x, y)=0,$$

is termed the 'scalar radical locus' of the abstract equation

$$\phi(x, y)=0,$$

and corresponds to what has been hitherto insufficiently designated as the 'locus of the equation' $\phi(x, y)=0$. It presents a necessarily imperfect image of that equation.

Clinant Plane Geometry.—Reverting to the original pair of equations

$$\mathrm{OP}=x\,.\,\mathrm{OI}+y\,.\,\mathrm{OJ},\quad \phi(x, y)=0,$$

and remembering that even if clinant ('impossible' or 'imaginary') values were substituted for x and y in the expressions $x\,.\,\mathrm{OI}$, $y\,.\,\mathrm{OJ}$, they would still represent definite lines (see abstract of Paper on 'Laws of Operation, &c.,' Proceedings, vol. x. p. 85), and consequently the line OP would still be perfectly determined, we see that the limitation of x and y to scalar values in the previous investigations was merely a matter of convenience. We may therefore give x any clinant value, and after determining the correspondent clinant value of y from $\phi(x, y)=0$ (which will always exist if the equation is algebraical), substitute these values in the concrete equation, and thus find OP, and consequently the locus of the equations. We may observe, however, that as a clinant involves *two* scalars, we must have some relation given between them, directly or indirectly, in order that there may be only one real variable, without which limitation the locus would in every case embrace the whole plane.

The general algebraical process is as follows, the Roman letter i being used for $+\sqrt{(-1)}$. Let $\mathrm{OP}=\mathrm{P}(\mathrm{X}_1, \mathrm{X}_2 \ldots \mathrm{X}_n)\,.\,\mathrm{OI}$, where $\mathrm{X}_1=p_1+\mathrm{i}\,.\,q_1, \ldots \mathrm{X}_n=p_n+\mathrm{i}\,.\,q_n$, and all the p, q are scalar. We can

reduce this expression to $OP=(P_1+i\,.\,P_2)\,.\,OI$, where P_1 and P_2 are scalar 'formations' (or 'functions') of the $2n$ scalars $p_1\,.\,.\,p_n$, $q_1\,.\,.\,q_n$. As this is equivalent to $OP=P_1\,.\,OI+P_2\,.\,OJ$, it is precisely the same as the general scalar concrete equation lately investigated. But one abstract equation will now no longer suffice; for if we put $P_1=x$, $P_2=y$, we must have $2n-1$ additional equations, in order ultimately to find $f(x, y)=0$ by eliminating $2n$ variables between $2n+1$ equations. The result, $OP=x\,.\,OI+y\,.\,OJ$, $f(x, y)=0$, with the conditions of scalarity, will then enable us to determine the locus by the usual process.

If there be only two clinants, $X=p+i\,.\,q$, and $Y=r+i\,.\,s$, and we have given $OP=P\,.\,(X, Y)\,.\,OI$; $C\,.\,(X, Y)=0$, where $C=0$ may be termed the 'curve equation,' these reduce to

$$OP=P_1\,.\,OI+P_2\,.\,OJ, \quad C_1=0, \quad C_2=0,$$

where P_1, P_2, C_1, C_2 are formations of p, q, r, s, so that, on putting $P_1=x$, $P_2=y$, we have only four equations, between which we cannot eliminate p, q, r, s. This again shows the necessity of some additional relation, $A(p, q, r, s)=0$, which may be called the 'assignant equation,' in order finally to discover $f(x, y)=0$, and thus determine the locus.

The only case ordinarily considered is where X is scalar. This corresponds to putting $q=0$ for the assignant equation. Hence q disappears and we have

$$OP=P_1\,.\,OI+P_2\,.\,OJ, \quad C_1=0, \quad C_2=0,$$

where P_1, P_2, C_1, C_2 are formations of p, r, s, and hence, putting $P_1=x$, $P_2=y$, we immediately eliminate p, r, s, and determine the locus.

This general theory is illustrated by numerous examples, and in particular Plücker's 'involutions' by means of 'imaginary lines' are fully explained by help of the really existent lines of this theory.

The general theory of the intersection of two 'clinant loci' is precisely analogous to that of two scalar loci with general concrete equations. In the particular case where the concrete equations are the same for both, or the reduced equations are

$$OP=[P_1(p, q, r, s)+i\,.\,P_2(p, q, r, s)]\,.\,OI,$$

$$C_1(p, q, r, s)=0,\ C_2(p, q, r, s)=0,\ A(p, q, r, s)=0,$$

and $OP'=[P_1(p', q', r', s')+i\,.\,P_2(p', q', r', s')]\,.\,OI$,

$$C_1'(p', q', r', s')=0,\ C_2'(p', q', r', s')=0,\ A'(p', q', r', s')=0,$$

the intersection will evidently be determined by putting $p=p'$, $q=q'$, $r=r'$, $s=s'$, which will give six simultaneous equations between four variables, p, q, r, s or p', q', r', s'. This gives two equations of condition. If, however, no assignant equations were given, we might determine the values of p, q, r, s from the four reduced curve equations, and then assume any assignant equations compatible with these solutions. This is more readily done by determining X and Y from the two unreduced curve equations $C(X, Y)=0$, $C'(X, Y)=0$. The process then corresponds to that for the simplest scalar case of intersection. If the values of X and Y prove to be scalar, then the assignant equations are $q=0$ and $s=0$, and we have an ordinary scalar case of intersection. But if this is not the case, and we find

$$X=a_1+\mathrm{i}\,.\,b_1\,\ldots\ldots\,a_n+\mathrm{i}\,.\,b_n;\ \ Y=c_1+\mathrm{i}\,.\,d_1\,\ldots\ldots\,c_n+\mathrm{i}\,.\,d_n,$$

we may take

$$(q-b_1)\ldots(q-b_n)=0,\quad (s-d_1)\ldots(s-d_n)=0,$$

among others, as assignant equations and determine the corresponding loci. These loci will be found to intersect in all the (perfectly real) points determined by the values of X and Y, but not necessarily in these only. Such points will of course not belong to the curves derived from putting $q=0$, $s=0$, and hence cannot in any sense be called points of intersection of these curves, although they have hitherto been termed 'imaginary points of intersection.'

The discovery of equations to loci described according to some geometrical law, furnishes a convenient illustration of this clinant theory. From any point O, draw radii vectores OU, OR to any curves, and make $RP=h\,.\,OU$, where h is scalar. Put

$$OP=(p+\mathrm{i}\,.\,q)\,.\,OI,\quad OR=(r+\mathrm{i}\,.\,s)\,.\,OI,\quad OU=(u+\mathrm{i}\,.\,v)\,.\,OI.$$

Then the condition $OP=OR+RP$ gives $p=r+hu$, $q=s+hv$, which correspond to the two reduced curve equations. We now require three more equations in order to eliminate four of the six scalars p, q, r, s, u, v, and find a relation between the two remaining scalars, sufficient with one of the above concrete equations to determine the locus of one of the points P, R, U. Two of these three equations will amount to assigning the locus of two of these points, and the third equation will amount to assigning some relation between the angular motions of OU and OR, without which the loci can clearly

not be determined. This third condition is frequently given in the form of requiring M (a point in RP, produced either way if necessary where $RM = k . OU$ and k is scalar) to lie on a given curve. It is then most convenient to introduce two new scalars m and n, so that $OM = (m + i . n) . OI$, when the condition $OM = OR + RM$, gives the two additional equations $m = r + ku$, $n = s + kv$, which with the five others will serve to eliminate six of the eight scalars, m, n, p, q, r, s, u, v, and leave the required abstract equation between the remaining two. The eliminations are very simple in a great variety of curves. This theory is fully illustrated by examples.

The first problem in the transformation of coordinates, 'given an alteration in the concrete and curve equations, to find the corresponding alteration in the assignant equation, so that the curves may remain identical, extent excepted,' is solved thus. Given

$$OP = (P_1 + i . P_2) . OI, \quad C_1 = 0, \quad C_2 = 0, \quad A = 0,$$

for the original curve, and

$$OP' = (P_1' + i . P_2') . OI, \quad C_1' = 0, \quad C_2' = 0$$

for the new equations, where the unaccented letters are formations of p, q, r, s, and the accented of p', q', r', s'. Since $OP = OP'$, we have $P_1 = P_1'$, $P_2 = P_2'$, between which and $C_1 = 0$, $C_2 = 0$, $A = 0$ eliminate p, q, r, s and use the final equation, which will only involve p', q', r', s' as the assignant equation $A' = 0$, which is independent of any particular form of the curve equations $C_1' = 0$, $C_2' = 0$. The ordinary case of the transformation of coordinates is a particular case of this. The second problem, 'given an alteration in the concrete and assignant equations, to find that in the curve equations,' requires the assignant equation to be put in the form $\phi + \psi = 0$, which is possible in an infinite number of ways. Then if the locus be that of

$$OP = x . OI + y . OJ, \quad \phi(x, y) + \psi(x, y) = 0,$$

and we have given

$$OP = [P_1(p, q, r, s) + i . P_2(p, q, r, s)] . OI,$$
$$\phi'(p, q, r, s) + \psi'(p, q, r, s) = 0 = A,$$

we put $\phi = \phi'$, $\psi = \psi'$, and hence finding

$$x = \xi(p, q, r, s), \quad y = \eta(p, q, r, s),$$

we use $P_1 = \xi$, $P_2 = \eta$ as the two curve equations. The third pro-

blem, 'given an alteration in the curve and assignant equations, to find that in the concrete equation,' admits of a similar solution. Given

$$OP = x \,.\, OI + y \,.\, OJ, \quad \phi(x, y) + \psi(x, y) = 0,$$

as the equations to the locus, and also

$C_1(p, q, r, s) = 0$, $C_2(p, q, r, s) = 0$, $A = \phi'(p, q, r, s) + \psi'(p, q, r, s =)0$

as the new curve and assignant equations: put $\phi = \phi'$, $\psi = \psi'$, and determine $x = \xi(p, q, r, s)$, $y = \eta(p, q, r, s)$, and then use

$$OP = (\xi + i \,.\, \eta) \,.\, OI$$

as the new concrete equation. The result is independent of the form of the curve equations. The geometrical significance of these transformations is that there are no 'families' of plane curves.

'Clinant radical loci,' or curves which furnish a sensible geometrical picture of the relations of the corresponding clinants which satisfy any abstract equation, may be obtained thus. Let the given equation be $C(X, Y) = 0$. This may be regarded as two reduced curve equations,

$$C_1(p, q, r, s) = 0, \quad C_2(p, q, r, s) = 0.$$

The values of X are assumed by drawing radii vctorees to points in any curve, and the corresponding values of Y have to be pictured. We must therefore have some equation $A = 0$, which in combination with the other two will give the curve by which X is thus determined. Eliminating the scalars, p, q, r, s, two and two, between these three equations, we obtain the following six, of which the first and last, and one of the intermediate ones, are in general only required:

$$f_1(p, q) = 0, \quad f_2(p, r) = 0, \quad f_3(p, s) = 0, \quad f_4(q, r) = 0,$$
$$f_5(q, s) = 0, \quad f_6(r, s) = 0.$$

We now construct the curves containing the points X, S, Y, as the loci of the equations

$$\begin{aligned} OX &= OP + PX = (p + i \,.\, q) \,.\, OI, & f_1(p, q) &= 0, \\ OS &= OP + PS = (p + i \,.\, s) \,.\, OI, & f_3(p, s) &= 0, \\ OY &= OR + RY = OR + PS = (r + i \,.\, s) \,.\, OI, & f_6(r, s) &= 0. \end{aligned}$$

Set off OP at pleasure on OI (produced both ways), and draw the ordinate PXS, cutting the two first curves in X and S. Through S draw SY parallel to OI, cutting the third locus in Y: then if $OX = X \,.\, OI$, and $OY = Y \,.\, OI$, X and Y are corresponding solutions

of $C(X, Y)=0$. When only scalar solutions are required, the abstract equations to the first and last curve become $q=0$, $s=0$, so that both of these curves coincide with OI produced both ways, and the intermediate or connecting curve is the ordinary scalar radical locus.

Scalar Solid Geometry.—The same theories apply with proper modifications. The concrete equation

$$OP = x \,.\, OI + y \,.\, OJ + z \,.\, OK,$$

with *one* abstract equation, $f(x, y, z)=0$, gives a surface for its locus, and with *two* abstract equations,

$$f_1(x, y, z)=0, \quad f_2(x, y, z)=0,$$

a curve. From these equations all the ordinary theories are most readily deduced. We may also take the concrete equation more generally in the form

$$OP = f_1(x, y, z) \,.\, OI + f_2(x, y, z) \,.\, OJ + f_3(x, y, z) \,.\, OK,$$

the abstract equation being $F(x, y, z)=0$. We obtain the surface by putting $x_1=f_1$, $y_1=f_2$, $z_1=f_3$, and eliminating x, y, z between these equations and $F=0$, thus finding $\phi(x_1, y_1, z_1)=0$. The locus of

$$OP = x_1 \,.\, OI + y_1 \,.\, OJ + z_1 \,.\, OK, \quad \phi(x_1, y_1, z_1)=0$$

must be limited by the condition that not only x, y, z, but also $f_1(x, y, z)$, $f_2(x, y, z)$, $f_3(x, y, z)$ must all be scalar. If three variable parameters a, b, c are introduced, we require in addition three equations of condition, $F_1=0$, $F_2=0$, $F_3=0$, between x, y, z, a, b, c, in order to eliminate all six and find $\phi(x_1, y_1, z_1)=0$.

Clinant Solid Geometry.—Some precaution is now necessary to indicate the *plane* on which the quadrantal rotation symbolized by i is to take place. This is effected by introducing all three coordinate axes into the concrete equations. Let the abstract equation be

$$F(X, Y, Z)=0.$$

Then, supposing

$$X=p+\mathrm{i}q, \quad Y=r+\mathrm{i}s, \quad Z=u+\mathrm{i}v,$$

this equation reduces to $F_1=0$, $F_2=0$, where F_1 and F_2 are formations of the six scalars p, q, r, s, u, v. We now require *two* assignant equations to determine the curves on which OX and OY are to be taken. These may be given in the form of the single clinant equa-

tion $A(X, Y, Z)=0$, which reduces to the two scalar equations $A_1=0$, $A_2=0$. Now take $OR=f_1(X, Y, Z) . OI+f_2(X, Y, Z) . OJ$. Then, since $i . OI=OJ$, because the assumption of the two axes determines the rotation to be in the plane IOJ, we can reduce this to $OR=R_1 . OI+R_2 . OJ$, where R_1, R_2 are formations of the six scalars.

By virtue of the equations $A_1=0$, $A_2=0$, $F_1=0$, $F_2=0$, which will give q, s, u, v in terms of p, r, the line OR is perfectly determined by the assumption of p and r. Next take

$$\sqrt{(R_1^2+R_2^2)} . OR_1=OR,$$

so that OR_1 is a unit radius in the direction OR. Put

$$OP=f_3(X, Y, Z) . OR_1+f_4(X, Y, Z) . OK,$$

which reduces to $OP=R_3 . OR_1+P_3 . OK$, where R_3, P_3 are formations of the six scalars, because $i . OR_1=OK$ on the plane $R_1 OK$. The locus of P will now manifestly be a surface, the concrete equation of which becomes of the usual form on putting for OR_1 its value. Determine P_1, P_2 by the equations

$$P_1 . \sqrt{(R_1^2+R_2^2)}=R_1 R_3, \quad P_2 . \sqrt{(R_1^2+R_2^2)}=R_2 R_3,$$

and we find

$$OP=P_1 . OI+P_2 . OJ+P_3 . OK.$$

Putting $x=P_1$, $y=P_2$, $z=P_3$, and eliminating the six scalars between these three equations and $A_1=0$, $A_2=0$, $F_1=0$, $F_2=0$, the locus becomes that of

$$OP=x . OI+y . OJ+z . OK, \quad \phi(x, y, z)=0,$$

which is limited by the conditions of scalarity.

After illustrating this theory by an example, the theory of intersection, when the elimination gives clinant values to determine the points, is discussed. The general theory is further illustrated by the determination of equations to loci, leading to very simple and extremely general modes of finding families of surfaces. The problems of the transformation of coordinates and of radical loci are shown to be precisely analogous to those of plane curves.

It will be evident that these investigations merely open out a new field for algebraical geometry, of which it is impossible to foresee the extent.